4/MTH–250 Syllabus–2023

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(May–June)

FYUP : 4th Semester Examination

MATHEMATICS

(Calculus—II)

(MTH–250)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* A function f is defined on the interval $[0, 1]$ as follows :

$$f(x) = \begin{cases} \dfrac{1}{2^n}, & \text{when } \dfrac{1}{2^{n+1}} < x \le \dfrac{1}{2^n} \ \forall\, n = 0, 1, 2, \ldots \\ 0, & \text{when } x = 0 \end{cases}$$

Show that f is integrable in $[0, 1]$ and find the value of $\int_0^1 f(x)\, dx$. 5

(b) Show that every continuous function on a closed and bounded interval is Riemann integrable. 4

(c) Show that a bounded function f is integrable on $[a, b]$ iff to each $\varepsilon > 0$, $\exists\, \delta > 0$ such that the oscillatory sum

$$|U(P, f) - L(P, f)| < \varepsilon$$

for all partitions P of $[a, b]$ with $||P|| < \delta$. 5

(d) Show that if f is bounded and integrable on $[a, b]$, then to every $\varepsilon > 0$, $\exists\, \delta > 0$ such that for every partition

$$P = \{a = x_0, x_1, \dots, x_n = b\}$$

of $[a, b]$ with norm $\le \delta$ and for every point $t_r \in [x_{r-1}, x_r]$

$$\left| \Sigma f(t_r)(x_r - x_{r-1}) - \int_a^b f \right| < \varepsilon$$

5

2. *(a)* Prove that the oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set of numbers

$$\{|f(\alpha) - f(\beta)| : \alpha, \beta \in [a, b]\}$$

5

(b) Show that if f is bounded and integrable in $[a, b]$, then $|f|$ is also bounded and integrable in $[a, b]$. 4

(c) If $\int_a^b f(x)\,dx$ exists and there exists a function ϕ such that $\phi'(x) = f(x)$, then show that

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

5

(d) State and prove the first mean value theorem of integral calculus. 1+4=5

UNIT—II

3. *(a)* Show that the sequence of functions $\{f_n\}$, where $f_n(x) = n/(x+n)$ is uniformly convergent in $[0, k]$, whatever k may be, but not uniformly convergent in $[0, \infty[$. 5

(b) Let $\{f_n\}$ be a sequence of derivable functions with pointwise limit f. Let f_n' be continuous for all n and let the sequence $\{f_n'\}$ be uniformly convergent with ϕ as uniform limit. Show that f is derivable and the derivative is equal to ϕ. 4

(c) State and prove Weierstrass' M-test for uniform convergence of a series of functions. 1+4=5

(d) Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x. 5

4. (a) Test the convergence of the integral

$$\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$$ 5

(b) Discuss the convergence of the improper integral

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx \ (m>0,\ n>0)$$ 5

(c) If ϕ is bounded and monotonic on $[a, \infty[$ and $\int_a^\infty f\,dx$ is convergent at ∞, then show that $\int_a^\infty f\phi\,dx$ is convergent at ∞. 5

(d) Show that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\left(\frac{b}{a}\right) \ (a, b>0)$$ 4

UNIT—III

5. (a) Show that

$$\int_C [(x-y)^3\,dx + (x-y)^3\,dy] = 3\pi a^4$$

taken along the circle $x^2+y^2=a^2$ in the counter-clockwise sense. 5

(b) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$. 3

(c) Change the order of integration in

$$\int_0^{2a}\int_{x^2/4a}^{3a-x}\phi(x, y)\,dx\,dy$$ 5

(d) If the double integral

$$\iint_R f(x, y)\,dx\,dy$$

exists where R is the rectangle $[a, b;\ c, d]$ and if $\int_a^b f(x, y)\,dx$ exists $\forall\, y \in [c, d]$, then show that the repeated integral

$$\int_c^d \left\{\int_a^b f(x, y)\,dx\right\} dy$$

exists and is equal to the double integral. 6

6. (a) Compute the volume enclosed by the surface $x^2+y^2=cz$, $x^2+y^2=2ax$, $z=0$. 7

(b) Evaluate

$$I = \iiint \sqrt{a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2}\, dx\,dy\,dz$$

taken throughout the domain

$$\left\{(x,\ y,\ z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$$ 7

(c) Evaluate

$$\iiint (x+y+z+1)^2\, dx\,dy\,dz$$

over the region defined by

$$x \ge 0,\ y \ge 0,\ z \ge 0,\ x+y+z \le 1$$ 5

UNIT—IV

7. (a) Show that

$$\int_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz = -2\pi ab^2$$

where the curve C is the part for which $x \ge 0$ of the intersection of the surfaces

$$x^2+y^2+z^2 = 2ax,\ x^2+y^2 = 2bx,\ a > b > 0$$

the curve begins at the origin and runs at first in the positive octant. 6

(b) Show that

$$I = \iint_S yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy = \frac{3}{8}$$

where S is the surface of the sphere $x^2+y^2+z^2 = 1$ in the first octant. 6

(c) Verify Green's theorem by evaluating it into two ways the line integral

$$\int_C x^2y\,dx + xy^2\,dy$$

taken along the closed path formed by $y = x$ and $x^2 = y^3$ in the first quadrant. 6

8. (a) Show that

$$\iint_S [(y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy] = a^3\pi$$

where S is the portion of the surface $x^2+y^2-2ax+az = 0,\ z \ge 0.$ 6

(b) Show using Gauss' theorem that the surface integral

$$\iint_S [(x^3-yz)\,dy\,dz - 2x^2y\,dz\,dx + 2\,dx\,dy]$$

taken over the outer surface of the cube bounded by the planes

$$x=0,\ x=a;\ y=0,\ y=a;\ z=0,\ z=a$$

is $\frac{1}{3}a^5$. 5

(c) State and prove Green's theorem. 1+6=7

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